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Impulsively started oscillations in a rotating stratified fluid

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The motion generated in an initially quiescent, incompressible, stratified, and/or rotating fluid of infinite extent when a spherical source begins to breathe fluid in and out periodically is considered. The properties of the resulting flow may be understood in terms of the inertial-internal waves which may propagate energy in the fluid. At all points located a finite distance from the source, except those points falling on certain conical surfaces which are tangent to the source and which contain the group velocity vector for waves at the source frequency, the flow is ultimately a steady oscillation at the source frequency. The manner in which the flow depends on source frequency is discussed in detail.

1. Introduction

We consider the motion generated in an initially quiescent, stratified, and/or rotating fluid of infinite extent when a spherical source of radius a begins to breathe fluid in and out with periodic normal velocity $\cos(\sigma t)$. The rotation vector and the direction of gravitational attraction are assumed antiparallel. The density field of the undisturbed fluid is supposed to be of the form $\exp(-\beta z)$ so that the Väisälä frequency, $N = (-(g/\rho_0)(\partial\rho_0/\partial z))^{\frac{1}{2}}$, is constant throughout the fluid. The stratification is weak in the Boussinesq sense; density variations are taken into account only in calculating buoyancy forces. Although the Boussinesq approximation may be locally valid everywhere in the fluid, it leads to a cumulative error in the solution of this problem at great vertical distances from the source. An attempt to solve the more general problem yields the present solution as a first approximation plus further finite terms multiplied by powers of βz. We shall therefore make the Boussinesq approximation from the outset and then apply the results only at vertical distances z from the source sufficiently small that $\beta z \ll 1$.

We focus attention on the flow long after the source has begun to pulsate. We follow the procedure of Stewartson (1952) very closely and are led to results similar to those obtained by Bretherton (1967) in his study of the development of a Taylor column in a rotating fluid. There are no advances in analysis beyond the work of Bretherton. The new results are a detailed description of the periodic flow and of the manner in which it varies as the frequency of the source is changed.

It is convenient to work in the cylindrical co-ordinates (r, ϕ, z) with velocity components (u, v, w), in order to take advantage of the axial symmetry which we assume. The geometry is sketched in figure 1.

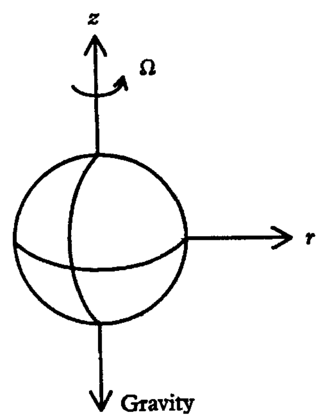

FIGURE 1.

2. Solution in integral form

Small oscillations of the fluid about a basic state of rest are governed by the following set of equations (in which all subscripts denote partial derivatives except for the 0 subscript on the density, which refers to the undisturbed density):

$$\left.\begin{aligned} u_t - fv &= -P'_r/\rho_0, \\ v_t + fu &= 0, \\ w_t &= -P'_z/\rho_0 - g\rho/\rho_0, \\ \rho_t + w\rho_{0z} &= 0 \end{aligned}\right\} \tag{1}$$

and

$$(1/r)(ru)_r + w_z = 0.$$

Here, f is the Coriolis parameter, P' the perturbation pressure, ρ_0 the undisturbed density and ρ the density perturbation. The boundary conditions are

$$\left.\begin{aligned} ru + zw = aU(t) \quad &\text{at} \quad r^2+z^2 = a^2 \\ \text{and} \qquad u, v, w, \rho \to 0 \quad &\text{as} \quad r, z \to \infty \quad \text{for fixed } t. \end{aligned}\right\} \tag{2}$$

Just after the source has begun to breathe fluid in and out, the motion should everywhere be that which would obtain if neither rotation nor stratification were present. The buoyancy force cannot be important during the initial motion because it depends on the vertical displacement of fluid particles for its effect and their initial displacement is zero even though their initial velocity does not vanish. A somewhat similar remark may be made about the initial effect of rotation (Morgan 1953). The initial motion (denoted by a tilde) thus satisfies

$$\tilde{u} = -\tilde{P}'_r/\rho_0, \quad \tilde{v} = 0, \quad \tilde{w} = -\tilde{P}'_z/\rho_0, \quad \tilde{\rho} = 0 \tag{3}$$

and

$$(1/r)(r\tilde{u})_r + \tilde{w}_z = 0, \tag{4}$$

with

$$r\tilde{u} + z\tilde{w} = aU(0) \quad \text{at} \quad r^2+z^2 = a^2. \tag{5}$$

We now use the solutions of (3) in a Laplace transformation of (1) and (2). If the transformed variables are defined by

$$\bar{w}(r,z,s) = \int_0^\infty e^{-st} w(r,z,t)\,dt, \tag{6}$$

with similar expressions for the remaining velocity components and for the pressure, we obtain

$$\left.\begin{aligned} s\bar{u}-f\bar{v} &= -\bar{P}_r,\\ s\bar{v}+f\bar{u} &= 0,\\ s\bar{w} &= -\bar{P}_z - g\rho/\rho_0,\\ s\bar{\rho}+\bar{w}\rho_{0z} &= 0,\end{aligned}\right\} \tag{7}$$

and $(1/r)(r\bar{u})_r+\bar{w}_z = 0,$

with $r\bar{u}+z\bar{w} = a\bar{U}(s)$ at $r^2+z^2=a^2$

and $\bar{u},\bar{v},\bar{w},\bar{\rho}$ finite as $r,z\to\infty$. (8)

The Boussinesq approximation has been used in passing from (1) to (7) by means of (3) and (6). Additionally, in (7) we have written

$$\bar{P} = \bar{P}'/\rho_0 - \tilde{P}/\rho_0.$$

From (7) we have

$$\bar{u} = -s\bar{P}_r/(s^2+f^2),\quad \bar{v} = f\bar{P}_r/(s^2+f^2),\quad \bar{w} = -s\bar{P}_z/(s^2+N^2). \tag{9}$$

These relations enable us to obtain a single boundary-value problem in $\bar{P}$ equivalent to (7) and (8), viz.

$$\frac{1}{r}(r\bar{P}_r)_r + \left(\frac{s^2+f^2}{s^2+N^2}\right)\bar{P}_{zz} = 0, \tag{10}$$

with

$$r\bar{P}_r + \left(\frac{s^2+f^2}{s^2+N^2}\right)(z\bar{P}_z) = -\left(\frac{s^2+f^2}{s}\right)a\bar{U}(s) \tag{11}$$

at $r^2+z^2=a^2$.

We may transform the differential equation (10) to Laplace's equation by appropriately scaling z

$$z' = z\left(\frac{s^2+N^2}{s^2+f^2}\right)^{\frac{1}{2}}.$$

The transformed boundary-value problem (10) and (11) becomes

$$\frac{1}{r}(r\bar{P}_r)_r + \bar{P}_{z'z'} = 0 \tag{12}$$

with

$$r\bar{P}_r + \left(\frac{s^2+f^2}{s^2+N^2}\right)(z'\bar{P}_{z'}) = -\left(\frac{s^2+f^2}{s}\right)a\bar{U}(s) \tag{13}$$

at

$$r^2 + [(s^2+f^2)/(s^2+N^2)]^{\frac{1}{2}} z'^2 = a^2.$$

The surface at which the boundary condition (13) is applied is a spheroid, oblate if $f^2 > N^2$ and prolate if $f^2 < N^2$. This suggests the introduction of the appropriate set of spheroidal co-ordinates in each case.

(i) When $f^2 > N^2$, we introduce oblate spheroidal co-ordinates (ξ,η) after Morse & Feshback (1953). The transformed Laplace equation is

$$[(\xi^2+1)\bar{P}_\xi]_\xi + [(1-\eta^2)\bar{P}_\eta]_\eta = 0, \tag{14}$$

with boundary condition

$$\overline{P}_{\xi} = -a\overline{U}(s)(s^2+N^2)^{\frac{1}{2}}(f^2-N^2)^{\frac{1}{2}}/s \tag{15}$$

at

$$\xi = [(s^2+N^2)/(f^2-N^2)]^{\frac{1}{2}}.$$

The co-ordinate transformations themselves are given by

$$r = a\left(\frac{f^2-N^2}{s^2+f^2}\right)^{\frac{1}{2}}(\xi^2+1)^{\frac{1}{2}}(1-\eta)^{\frac{1}{2}}, \quad z = a\left(\frac{f^2-N^2}{s^2+N^2}\right)^{\frac{1}{2}}\xi\eta. \tag{16}$$

Note that if $N = 0$, these reduce to the co-ordinate transformations of Stewartson (1952).

The solution of (14) subject to boundary condition (15) is independent of η and is easily seen to be

$$\overline{P}(\xi) = \frac{-ia}{2}\overline{U}(s)\left(\frac{s^2+f^2}{s}\right)\left(\frac{s^2+N^2}{f^2-N^2}\right)^{\frac{1}{2}}\log\left(\frac{\xi+i}{\xi-i}\right). \tag{17}$$

By making use of (9), we find

$$w(r,z,t) = \frac{a}{2\pi i}\int_{-\infty i}^{+\infty i} e^{st}ds\,\frac{\overline{U}(s)(s^2+f^2)}{(s^2+N^2)^{\frac{1}{2}}(f^2-N^2)^{\frac{1}{2}}}\frac{\xi_z}{\xi^2+1}. \tag{18}$$

Some manipulation allows us to write

$$\frac{\xi\xi_z}{\xi^2+1} = \frac{\pm(s^2+N^2)z}{(r^2+z^2)(s^2+\Sigma_+^2)^{\frac{1}{2}}(s^2+\Sigma_-^2)^{\frac{1}{2}}}, \tag{19}$$

where

$$2a^2(f^2-N^2)\xi^2 = \{(s^2+f^2)r^2+(s^2+N^2)z^2-(f^2-N^2)a^2\} \pm[\{\ \}^2+4a^2(f^2-N^2)(s^2+N^2)]^{\frac{1}{2}} \tag{20}$$

and where we have introduced

$$\Sigma_{\pm}^2 \equiv f^2\left(\frac{r(r^2+z^2-a^2)^{\frac{1}{2}}\pm az}{r^2+z^2}\right)^2+N^2\left(\frac{z(r^2+z^2-a^2)^{\frac{1}{2}}\mp ar}{r^2+z^2}\right)^2. \tag{21}$$

The undetermined signs in (19) and (20) are identical and we show subsequently that the plus sign must be chosen. The significance of the frequencies Σ_+ and Σ_- will be explained when we evaluate the integrals for u, v and w.

We therefore have for w the following integral expression

$$w(r,z,t) = \frac{az}{(2\pi i)(r^2+z^2)(f^2-N^2)^{\frac{1}{2}}}\int_{-\infty i}^{+\infty i} e^{st}ds\,\frac{\overline{U}(s)(s^2+f^2)(s^2+N^2)^{\frac{1}{2}}}{(s^2+\Sigma_+^2)^{\frac{1}{2}}(s^2+\Sigma_-^2)^{\frac{1}{2}}}\left(\frac{1}{\xi(s)}\right), \tag{22}$$

with results for u and v obtained in a similar manner

$$u(r,z,t) = \frac{ar}{(2\pi i)(r^2+z^2)(f^2-N^2)^{\frac{1}{2}}}\int_{-\infty i}^{+\infty i} e^{st}ds\,\frac{\overline{U}(s)(s^2+f^2)(s^2+N^2)^{\frac{1}{2}}}{(s^2+\Sigma_+^2)^{\frac{1}{2}}(s^2+\Sigma_-^2)^{\frac{1}{2}}}\left(\frac{\xi}{\xi^2+1}\right) \tag{23}$$

and

$$v(r,z,t) = \frac{-arf}{(2\pi i)(r^2+z^2)(f^2-N^2)^{\frac{1}{2}}}\int_{-\infty i}^{+\infty i} \frac{e^{st}ds}{s}\,\frac{\overline{U}(s)(s^2+f^2)(s^2+N^2)^{\frac{1}{2}}}{(s^2+\Sigma_+^2)^{\frac{1}{2}}(s^2+\Sigma_-^2)^{\frac{1}{2}}}\left(\frac{\xi}{\xi^2+1}\right). \tag{24}$$

(ii) When $f^2 < N^2$, we introduce prolate spheroidal co-ordinates (ξ, η), again after Morse & Feshbach (1953). The transformed Laplace equation is

$$[(\xi^2-1)\bar{P}_\xi]_\xi + [(1-\eta^2)\bar{P}_\eta]_\eta = 0, \tag{25}$$

with boundary conditions

$$\bar{P}_\xi = -a\bar{U}(s)(s^2+N^2)^{\frac{1}{2}}(N^2-f^2)^{\frac{1}{2}}/s \tag{26}$$

at

$$\xi = [(s^2+N^2)/(N^2-f^2)]^{\frac{1}{2}}.$$

The co-ordinate transformations themselves are given by

$$r = a\left(\frac{N^2-f^2}{s^2+f^2}\right)^{\frac{1}{2}}(\xi^2-1)^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}, \quad z = a\left(\frac{N^2-f^2}{s^2+N^2}\right)\xi\eta. \tag{27}$$

The solution of (26) subject to (27) again has no η dependence and may be written in the form

$$\bar{P}(\xi) = \tfrac{1}{2}a\bar{U}(s)\frac{(s^2+f^2)}{s}\left(\frac{s^2+N^2}{N^2-f^2}\right)^{\frac{1}{2}}\log\left(\frac{1+\xi}{1-\xi}\right). \tag{28}$$

We now proceed exactly as when $f^2 > N^2$ and find expressions for u, v and w. They are identical with (22), (23) and (24) except that the factor (f^2-N^2) which appears in these integrals and in $\xi(s)$ given by (21) must be replaced by (N^2-f^2), and the factor ξ^2+1 appearing in (23) and (24) must be replaced by ξ^2-1. For brevity we shall discuss only the case $f^2 > N^2$. In both cases the surface of constant ξ corresponding to the surface of the source is $\xi = ((s^2+N^2)/|f^2-N^2|)^{1/2}$ provided that the proper choice of sign is made in (21). It is easily verified, using this result, that (22) and (23) together satisfy boundary condition (2) at the source.

3. Infinitesmal free waves

The flow at long times after the source has begun to pulsate may best be understood in terms of the inertial-internal waves which may propagate in a rotating and/or stratified fluid. Since Eckart (1960), Phillips (1963) and Sandstrom (1966), as well as many others, have investigated the properties of these waves, we here simply point out their salient features in our notation. The frequency σ_θ of waves of any wavelength propagating *energy* at an angle θ with the vertical is given by

$$\sigma_\theta^2 = f^2\sin^2\theta + N^2\cos^2\theta. \tag{29}$$

The phase velocity is

$$\mathbf{C}(\mathbf{k}) = (\mathbf{k}/|k|)(\sigma_\theta/|k|), \tag{30}$$

whereas the group velocity of energy propagation is

$$\mathbf{C}_g(\mathbf{k}) = \frac{\mathbf{k}}{|k|}\frac{(N^2-f^2)\sin^2\theta}{|k|\sigma_\theta} - \hat{\mathbf{z}}\frac{(N^2-f^2)\sin\theta}{|k|\sigma_\theta} \tag{31}$$

($\hat{\mathbf{z}}$ is a unit vector along the positive z axis).

Clearly $\mathbf{C}\,.\,\mathbf{C}_g = 0$. If energy is being radiated away from a source, the progression of phases is towards the axis of rotation if $N^2 < f^2$ and away from it if $N^2 > f^2$. From (30) and (31), the magnitude of the group velocity vector may be shown to be

$$|\mathbf{C}_g(\mathbf{k})| = \frac{|N^2-f^2|\,|\sin 2\theta|}{2|k|\sigma_\theta}. \tag{32}$$

If the fluid is stratified but not rotating, (31) indicates that energy at very low frequencies propagates in a nearly horizontal direction, whereas if the fluid is rotating but not stratified, (31) indicates that the energy at very low frequencies propagates in a nearly vertical direction. In two dimensions, Bretherton (1967) has shown that this ability of the waves to transmit energy at very low frequencies is sufficient to account for the formation of Taylor columns in a rotating fluid or their analogues in a stratified fluid. If both rotation and stratification are present simultaneously, it follows from (29) and (31) that waves propagating energy may exist only for frequencies between, but not including, N and f. In particular, the possibility of Taylor columns at zero frequency is then lost. Thus, a source oscillating at some frequency between N and f may radiate energy via internal-inertial waves. But motions generated at frequencies N or f or at any frequency outside of these limits will be of the evanescent sort, trapped around the generator and carrying no energy away.

4. Details of the solution

It is convenient to work in terms of the distance $R = (r^2+z^2)^{\frac{1}{2}}$ from point (r,z) to the centre of the sphere and the colatitude $\theta = \operatorname{arc}\cos(z/R)$ of point (r,z).

The integrals (22), (23) and (24) with definitions (20) and (21) describe the inviscid motion at all times. We shall take

$$U(t) = \cos(\sigma t), \quad \overline{U}(s) = s/(s^2+\sigma^2) \tag{33}$$

and examine the motion after many pulsations of the source.

We have, from (20) and (21),

$$2a^2(f^2-N^2)\xi^2 = \tfrac{1}{2}R^2[(s^2+\Sigma_+^2)^{\frac{1}{2}}+(s^2+\Sigma_-^2)^{\frac{1}{2}}]^2 - 2a^2(f^2-N^2)\cos^2\theta. \tag{34}$$

This suggests a useful approximation as $R/a \to \infty$ for most s. We choose these square roots to be positive real as $s \to 0$ and, therefore, likewise choose the square root in (21) to be positive real as $s \to 0$. With this choice of branch, only the choice of the plus sign in (21) and hence in (34) as well will make ξ as given in (34) reduce to the proper function of s, given in (15) on the surface of the sphere. With this choice of sign, $\xi(s)$ is uniquely defined.

From (21), unless $z = 0$, $\xi^2(s)$ vanishes only if $s = \pm iN$. If $s = \pm iN$, we have

$$\left.\begin{aligned} &\xi^2(\pm iN) = 0 && \text{if} \quad r < a \\ \text{and} \quad &\xi^2(\pm iN) = (r^2-a^2)/a^2 && \text{if} \quad r > a. \end{aligned}\right\} \tag{35}$$

Similarly, unless $r = 0$, $\xi^2(s)+1$ may vanish only if $s = \pm if$. If $s = \pm if$, we have

$$\left.\begin{aligned} &\xi^2(\pm if)+1 = 0 && \text{if} \quad z < a \\ \text{and} \quad &\xi^2(\pm if)+1 = (z^2-a^2)/z^2 && \text{if} \quad z > a. \end{aligned}\right\} \tag{36}$$

The integrands of (22), (23) and (24) thus have simple poles at $s = \pm i\sigma$ with branch points at $s = \pm i\Sigma_+$, $s = \pm i\Sigma_-$ and possibly at $s = \pm iN$, $s = \pm if$. For $r < a$, ξ^2 has simple zeros at $s = \pm iN$ so that the integrand of (22) is regular at $s = \pm iN$ if $r < a$. If $r > a$, ξ^2 is not zero at $s = \pm iN$ and the integrand goes as $(s^2+N^2)^{\frac{1}{2}}$ near $s = \pm iN$. If $r = a$, ξ^2 goes as $(s^2+N^2)^{\frac{1}{2}}$ near $s = \pm iN$ so that the integrand goes as $(s^2+N^2)^{\frac{1}{2}}$ there. Similarly, if $z < a$, ξ^2+1 goes as s^2+f^2 near $s = \pm if$ so that the integrands of (23) and (24) are always regular there.

From (34), $|\xi| \to \infty$ as $R \to \infty$ for most s so that the factor $\xi/(\xi^2+1)$ appearing in (23) and (24) is well approximated by $1/\xi$ for large R/a. This replacement in (23) leads to

$$u(r,z,t)/\sin\theta = w(r,z,t)/\cos\theta \quad \text{if} \quad R \gg a, \tag{37}$$

i.e. to lowest order in a/R, u and w are simply the components of a radial velocity vector. This approximation suggests the nature of the flow very far from the source but it is not sufficient for all purposes. In particular, a further term will be found necessary to conserve mass in the most intense portion of the flow.

Some simple geometry shows that the frequencies $\Sigma_{\pm}(r,z)$ are the frequencies of waves which propagate energy towards the point (r,z) along the uppermost and lowermost tangents from the point (r,z) to the spherical source. This is most easily seen by computing from (29), to lowest order in a/R, the square of the frequency at which waves coming from these upper and lower tangents towards the point of observation $(R,\theta) = (r,z)$ would propagate energy. The result is, to lowest order in a/R, $\Sigma^2_{\pm}(r,z)$ as defined in (20). Some manipulation shows that the correspondence is valid for all values of R and not only for large R. Some convenient approximations for these frequencies $\Sigma^2_{\pm}(R,\theta)$, all derived directly from the definition (20), are:

$$\left.\begin{aligned}
&\Sigma^2_{\pm} = \sigma^2_{\theta} \pm (a/R)(f^2-N^2)\sin 2\theta && \text{if} \quad R \gg a,\\
&\Sigma^2_{+} = \Sigma^2_{-} = f^2a^2/z^2+N^2(1-a^2/z^2) && \text{if} \quad \theta = 0,\\
&\Sigma^2_{+} = \Sigma^2_{-} = f^2(1-a^2/r^2)+N^2a^2/r^2 && \text{if} \quad \theta = \tfrac{1}{2}\pi,\\
&\Sigma^2_{-} = N^2 && \text{if} \quad r = a,\\
&\Sigma^2_{+} = f^2 && \text{if} \quad z = \pm a\\
\text{and} \quad &\Sigma^2_{+} = \Sigma^2_{-} = f^2\cos^2\theta+N^2\sin^2\theta && \text{if} \quad R = a.
\end{aligned}\right\} \tag{38}$$

Notice that the two frequencies become equal on the sphere, very far from it, directly above its centre, or in its equatorial plane. For large R, $\Sigma_{\pm}(R,\theta)$ closely approximates σ_{θ} given in (29).

5. Transient motion

The initial transient contains energy at all frequencies and we expect to see that part of the transient energy having a frequency between N and f leaving the source along the appropriate cones of constant θ given by (29). The propagation of this part of the energy of the transient away from the source has its mathematical expression in the occurrence and nature of the singularities at $s = \pm i\Sigma_{+}$ and $s = \pm i\Sigma_{-}$ in the integrands of (22), (23) and (24). At a location $(R \gg a, \theta)$ not near the cone along which energy will be propagated away from the source at its frequency σ of pulsation, we may approximate the transient motion by neglecting the variation of all functions of s in the integrand of (22) except those contributing to the branch points at $s = \pm i\Sigma_{+}$ and $s = \pm i\Sigma_{-}$ to find

$$w(R \gg a, \theta, t \gg \sigma_{\theta}^{-1}) = \frac{a\cos^2\theta\,(f^2-N^2)^{\frac{1}{2}}\sigma_{\theta}}{R(\sigma^2-\sigma^2_{\theta})}[J_0(\Sigma_{+}t)-J_0(\Sigma_{-}t)]. \tag{39}$$

The transient disturbance far from the sphere appears as two wave trains, one from the upper tangent to the sphere and one from the lower. At times small enough that

$$(\Sigma_+ t - \Sigma_- t) = \frac{a(f_-^2 - N^2)\sin 2\theta}{R\sigma_\theta} t \ll \pi,$$

but still large enough that $\Sigma_\pm t \gg 1$, the radial velocity goes as $t^{\frac{1}{2}} \sin(\sigma_\theta t - \pi/4)$, i.e. it is initially a growing oscillation at frequency σ_θ. From the magnitude (32) of the group velocity of energy propagation, it is evident that this corresponds to times sufficiently small that waves of wavelength $4a$ and shorter have not yet arrived. Only after the longest of these have arrived does the transient begin to decay, finally going as

$$t^{-\frac{1}{2}} \sin(\sigma_\theta t - \pi/4) \sin[(a(f^2 - N^2)\sin 2\theta/R\sigma_\theta)t],$$

the result of two interfering wave trains.

Since, for large R, $\Sigma_\pm(R,\theta)$ are closely approximated by the frequency σ_θ, which, when $f^2 > N^2$, is an increasing function of θ, lines of constant phase are lines of constant θ and these move towards the axis of rotation. The transient therefore contain only waves which transmit energy *away* from the source. When $f^2 < N^2$, σ_θ is a decreasing function of θ so that phases move away from the axis of rotation and again only waves carrying energy away from the source are present in the transient.

This description of the transient requires modification near $\theta = 0$ where $\Sigma_\pm \simeq N$, and near $\theta = \frac{1}{2}\pi$, where $\Sigma_\pm \simeq f$.

6. Flow at the source frequency

At any point (R,θ), the development of the flow which is harmonic at the frequency σ of the source may be expressed as a superposition of transients emitted from the source at successive times. The motion may be approximated by neglecting the variation of all functions of s in the integral except for the poles at $s = \pm i\sigma$ and the nearby branch points at $s = \pm i\Sigma_+$, and $s = \pm i\Sigma$. The result is, by the convolution theorem for Laplace transforms

$$w(R \gg a, \theta, t \gg \sigma^{-1}) = \frac{a(f^2-\sigma^2)(\sigma^2-N^2)^{\frac{1}{2}}}{R(f^2-N^2)(2\sin\theta)} \int_0^t \sin[\sigma(t-t')][J_0(\Sigma_+ t') - J_0(\Sigma_- t')]dt'. \tag{40}$$

The structure of the vertical velocity field is most easily seen in the limit $t \to \infty$, where the integration may be carried out exactly to yield

$$w(R \gg a, \theta, t \gg \sigma^{-1}) = \frac{a(f^2-\sigma^2)(\sigma^2-N^2)^{\frac{1}{2}}}{R(f^2-N^2)(2\sin\theta)} \begin{Bmatrix} \left[\dfrac{\sin\sigma t}{(\Sigma_+^2-\sigma^2)^{\frac{1}{2}}} - \dfrac{\sin\sigma t}{(\Sigma_-^2-\sigma^2)^{\frac{1}{2}}}\right] (\Sigma_+^2 > \Sigma_-^2 > \sigma^2), \\ \left[\dfrac{\sin\sigma t}{(\Sigma_+^2-\sigma^2)^{\frac{1}{2}}} + \dfrac{\cos\sigma t}{(\sigma^2-\Sigma_-^2)^{\frac{1}{2}}}\right] (\Sigma_+^2 > \sigma^2 > \Sigma_-^2), \\ \left[\dfrac{\cos\sigma t}{(\sigma^2-\Sigma_+^2)^{\frac{1}{2}}} + \dfrac{\cos\sigma t}{(\sigma^2-\Sigma_-^2)^{\frac{1}{2}}}\right] (\sigma^2 > \Sigma_+^2 > \Sigma_-^2), \end{Bmatrix} \tag{41}$$

a result which may also be obtained directly as the sum of the residues at $s = \pm i\sigma$ in (22).

Similar expressions may likewise be obtained for the other velocity components. The phase and variation with position of the velocity field are quite different in the different regions indicated in (41). The region for which $\Sigma_+^2 > \sigma^2 > \Sigma_-^2$ lies between the two parallel cones of constant $\theta = \theta_\sigma$ tangent to the sphere above and below (see figure 2). There is a rapid variation of the phase of the motion across this region and the vertical velocity as given by (41) grows without limit as we approach the edges of the region. Points poleward of this are contained in the region for which $\sigma^2 > \Sigma_+^2 > \Sigma_-^2$ and those equatorward of it are contained in the region for which $\Sigma_+^2 > \Sigma_-^2 > \sigma^2$. These regions are sketched in figure 2. As $R \to \infty$ along any line of constant θ other than the one for which $\theta = \theta_\sigma$, we enter the second and third regions. Σ_+ and Σ_- approach one another as $R \to \infty$ and the radial variation of the vertical velocity is R^{-2}. As $R \to \infty$ along the cone of constant $\theta = \theta_\sigma$ both Σ^+ and Σ^- approach σ itself and the radial variation of the vertical velocity is $R^{-\frac{1}{2}}$.

To this order of approximation in a/R, the velocity field given by (41) and (37) does not display the radial variation necessary to conserve mass within the region of greatest velocities. There must therefore be an appreciable flow normal to the cones of constant $\theta = \theta_\sigma$ along which energy at frequency σ flows. If we retain the next order in a/R in summing the residues in (23) and write the result in terms of ϵ, the normal distance from the point of observation to the line $\theta = \theta_\sigma$ passing through the centre of the sphere, we obtain a flow having radial velocity going as $(a/R)^{\frac{1}{2}}$ plus a normal component going as $(a/R)^{\frac{3}{2}}$:

$$u_{\text{radial}} = \left(\frac{a}{R}\right)^{\frac{1}{2}} \frac{(f^2-\sigma^2)(\sigma^2-N^2)^{\frac{1}{2}}}{(f^2-N^2)^{\frac{3}{2}}\sin(2\theta_\sigma)^{\frac{3}{2}}} \begin{cases} \left[\dfrac{-\sin\sigma t}{(\epsilon/a-1)^{\frac{1}{2}}} + \dfrac{\sin\sigma t}{(\epsilon/a+1)^{\frac{1}{2}}}\right] & (\epsilon > a), \\ \left[\dfrac{\cos\sigma t}{(1-\epsilon/a)^{\frac{1}{2}}} + \dfrac{\sin\sigma t}{(\epsilon/a+1)^{\frac{1}{2}}}\right] & (a > \epsilon > -a), \\ \left[\dfrac{\cos\sigma t}{(1-\epsilon/a)^{\frac{1}{2}}} - \dfrac{\cos\sigma t}{(-1-\epsilon/a)^{\frac{1}{2}}}\right] & (-a > \epsilon) \end{cases}$$

and

$$u_{\text{normal}} = \left(\frac{a}{R}\right)^{\frac{3}{2}} \frac{(f^2-\sigma^2)(\sigma^2-N^2)^{\frac{1}{2}}}{(f^2-N^2)^{\frac{3}{2}}(\sin 2\theta_\sigma)^{\frac{3}{2}}} \begin{cases} -(\epsilon/a-1)^{\frac{1}{2}}\sin\sigma t + (\epsilon/a+1)\sin\sigma t & (\epsilon > a), \\ -(1-\epsilon/a)^{\frac{1}{2}}\cos\sigma t + (\epsilon/a+1)^{\frac{1}{2}}\sin\sigma t & (a > \epsilon > -a), \\ -(1-\epsilon/a)^{\frac{1}{2}}\cos\sigma t - (-1-\epsilon/a)^{\frac{1}{2}}\cos\sigma t & (-a > \epsilon). \end{cases} \quad (42)$$

At a large radial distance from the source, the persistent forced motion is almost entirely confined between the two parallels, tangent to the source above and below, whose direction is the direction of energy flow at the source frequency. Along these two parallels, the radial flow of (42) has become singular as $t \to \infty$ for fixed R. For finite times, this singularity does not arise. Along the parallel $\Sigma_+ = \sigma$ we have from (40)

$$w \simeq \frac{a(f^2-\sigma^2)(\sigma^2-N^2)^{\frac{1}{2}}}{R(f^2-N^2)(2\sin\theta_\sigma)}[tJ_1(\sigma t)], \quad (43)$$

which shows that w ultimately grows as $t^{\frac{1}{2}}$. The same holds along the parallel $\Sigma_- = \sigma$.

The flow at frequency σ is thus ultimately steady except on those cones $\Sigma_\pm = \sigma$ along which the source continuously radiates energy. It is approached as a superposition of waves arriving from the upper and lower edge of the source, all carrying energy away from the source. At any time along the cones $\Sigma_\pm = \sigma$, only waves having a certain minimum wavelength have yet arrived so that the amplitude of the oscillation grows without limit.

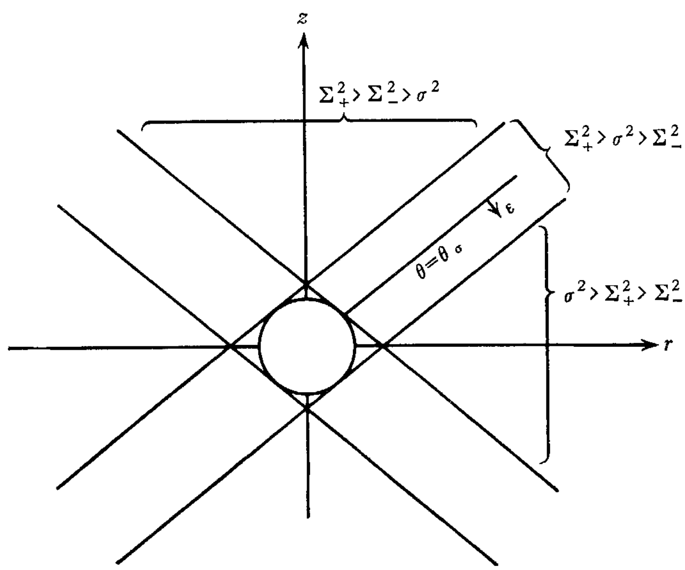

FIGURE 2.

Of course, the foregoing procedure must be modified whenever the point of observation (R, θ) is within or near the vertical cone $r = a$ circumscribing the source and whenever (R, θ) is between or near the horizontal planes $z = \pm a$ tangent to the source at its poles because then one of the frequencies Σ_+ is very close to N or to f.

7. The oscillation at the Väisälä frequency

Some of the transient energy goes into a decaying oscillation at the Väisälä frequency itself. The details of how this component of the motion builds up must be sought by using approximations appropriate at times earlier than those for which the present techniques are valid, but its spatial structure as $t \to \infty$, the contribution to (20), (21) and (22) from the branch points of the integrals at $s = \pm iN$, is in agreement with what we expect from the nature of the equation governing harmonic motion, according to which the motion at the Väisälä frequency in a fluid in which the Väisälä frequency N shows a striking, if somewhat superficial, resemblance to the Taylor column in the case of pure rotation. Any harmonically varying dynamical variable satisfies an equation of the form

$$w_{zz} - \left(\frac{N^2-\sigma^2}{\sigma^2-f^2}\right)(w_{xx}+w_{yy}) = 0. \tag{44}$$

From this, if $\sigma = N$, the fields may vary at most linearly with z. In the present case, in which we expect finite velocities far from the source, there can be no vertical variation of flow at the Väisälä frequency.

This means that if no oscillation at the Väisälä frequency is imposed at the source, then there can be none within the cylinder $r < a$ circumscribing it. For $r > a$, only a decaying transient oscillation at the Väisälä frequency is possible. Accordingly, the integrand of (22) is regular near $s = \pm iN$ for $r < a$ and behaves like $(s^2+N^2)^{\frac{1}{2}}$ near $s = \pm iN$ for $r > a$. When $r > a$, there is therefore a transient oscillation at the Väisälä frequency which decays like $t^{-\frac{3}{2}}$. For $r < a$, this transient does not appear.

8. Oscillations driven at the Väisälä frequency or at the inertial frequency

If the source frequency σ is equal to the Väisälä frequency N, then, by the arguments of the foregoing section, we expect the persistent motion to be ultimately confined within the cylinder $r < a$. If $\sigma = N$, the contribution to (22) from the neighbourhood of $s = \pm iN$ is given by

$$\left.\begin{aligned} w &= (a/(a^2-r^2)^{\frac{1}{2}})\cos Nt \quad &\text{if} \quad r < a \\ \text{and} \qquad w &\propto t^{-\frac{1}{2}} &\text{if} \quad r > a. \end{aligned}\right\} \tag{45}$$

For $r < a$, the integrand of (23) is regular near $s = \pm iN$ and, for $r > a$, it varies as $(s^2+N^2)^{\frac{1}{2}}$ near $s = \pm iN$. There is thus a transient radial motion at the Väisälä frequency only when $r > a$. The flow within $r < a$ is made up of waves of vanishing group velocity and is thus an evanescent motion, exactly in phase with the source and carrying no energy away from it. If $N = 0$, the first of (45) is Stewartson's (1953) result for the vertical velocity in a Taylor column above a spherical source. Stewartson also found a persistent swirling velocity v outside of the Taylor column in the case of zero stratification, but when oscillations at a non-zero Väisälä frequency N are considered, this component ultimately decays as $t^{-\frac{1}{2}}$.

If the source frequency σ is equal to the inertial frequency f, we similarly expect the motion ultimately to be confined to $-a < z < a$. In this case, (22) and (23) yield $u \propto \cos \sigma t$ with no persistent vertical motion for $|z| < a$ and no persistent velocities elsewhere. Again, the motion is of the evanescent sort.

9. Other features of the motion

At certain locations in space this inviscid calculation produces persistent oscillations at frequencies other than σ. These are, of course, artifacts of the inviscid world, but they are always understandable in terms of the properties of inertial-internal waves.

Exactly above the source, at $\theta = 0$, the frequencies $\Sigma_{\pm}$ of the two wave trains emitted tangentially from the source are equal and the result is a persistent oscillation at frequency $(f^2a^2/z^2+N^2(1-a^2/z^2))^{\frac{1}{2}}$. The waves making up this oscillation propagate their energy nearly vertically so that their group velocity nearly vanishes. This makes the motion at $\theta = 0$ very slow in approaching any final steady amplitude even though there is no continuing flux of energy out of the source in this direction. A very similar state of affairs prevails in the equatorial plane of the source, resulting in a persistent oscillation at frequency

$$(f^2(1-a^2/r^2)+N^2a^2/r^2)^{\frac{1}{2}}.$$

On the source, a residue of waves of infinitesimal length which never propagate their energy away from the source results in a persistent oscillation at

$$(f^2\cos^2\theta+N^2\sin^2\theta)^{\frac{1}{2}}.$$

Finally at $r = a$, on the cylinder circumscribing the source, and at $z = \pm a$, on the two horizontal planes tangent to the source at its poles, the decay of the transient motion is altered because one of the transient wave trains has the Väisälä frequency or the inertial frequency and so does not transport energy away from the source.

The action of viscosity will, of course, damp all of these oscillations and will generally wipe out all features of the inviscid motion which are due to the existence of vanishingly short waves carrying energy with vanishingly small speed. In particular, the flow driven at the source frequency will not ultimately become singular along the two tangent cones parallel to the direction of energy flow. It is easy in principle to extend Bretherton's approximate allowance for viscosity in the structure of the Taylor column to the present case, but we shall omit the details here.

10. Motion at various source frequencies

The Taylor column phenomenon is associated with those waves which propagate energy at vanishingly small frequencies. If the fluid rotates but is not stratified, energy being carried along a cone of constant θ is associated with waves of frequency $\sigma = f\sin(\theta)$. By (31), the velocity of energy propagation vanishes when $\sigma = f$ but is non-zero and vertical when $\sigma = 0$. If the fluid is stratified but does not rotate, energy being carried along cones of constant θ is associated with waves of frequency $\sigma = N\cos(\theta)$. By (31), the velocity of energy propagation vanishes when $\sigma = N$ but is non-zero and horizontal when $\sigma = 0$. But if both rotation and stratification are present simultaneously, then energy being carried along cones of constant θ is associated with waves of frequency

$$\sigma = (N^2\cos^2\theta+f^2\sin^2\theta)^{\frac{1}{2}}.$$

By (31), the velocity of energy propagation is non-zero only for σ between (but not equal to either of) f and N. The possibility of a Taylor column in the traditional sense is then lost and the slow motion of the fluid differs greatly from that which obtains when either N or f vanish separately.

If the source oscillates at a frequency outside of the range f to N, the entire wave-like nature of the motion is lost. The governing equation (44) is elliptic and reduces to Laplace's equation if the vertical co-ordinate is rescaled by the real factor $((\sigma^2 - N^2)/(\sigma^2 - f^2))^{\frac{1}{2}}$. The boundary condition corresponding to (13) is then to be applied at the spheroid $r^2 + z'^2(\sigma^2 - f^2)/(\sigma^2 - N^2) = a^2$. The solution, the potential flow exterior to a spheroidal source in a perfect fluid, becomes the desired solution of (44) with appropriate boundary conditions when rewritten in the unscaled vertical co-ordinate. Direct evaluation of the residues at $s = \pm i\sigma$ in (22), (23) and (24) shows that for large R/a and large σ,

$$\left.\begin{aligned} w &= \frac{a^2 \cos\theta}{R^2} \frac{(\sigma^2 - f^2)(\sigma^2 - N^2)^{\frac{1}{2}}}{(\sigma^2 - \sigma_\theta^2)^{\frac{3}{2}}} \cos \sigma t, \\ u &= \frac{a^2 \sin\theta}{R^2} \frac{(\sigma^2 - f^2)(\sigma^2 - N^2)^{\frac{1}{2}}}{(\sigma^2 - \sigma_\theta^2)^{\frac{3}{2}}} \cos \sigma t \\ \text{and} \quad v &= (f/\sigma^2)\, u_t. \end{aligned}\right\} \tag{46}$$

If $\sigma \gg f$ and $\sigma \gg N$, this reduces to the flow exterior to a spherical source in a perfect fluid, as the previous reasoning indicates that it must. If $f = 0$, we have

$$w = \frac{a^2 \cos\theta}{R^2} \frac{\sigma^2(\sigma^2 - N^2)^{\frac{1}{2}}}{(\sigma^2 - N^2 \cos^2\theta)^{\frac{3}{2}}} \cos \sigma t.$$

This clearly vanishes as σ approaches N from above unless $\theta = 0$, when it grows without limit. As σ approaches N, the flow field, spherically symmetric for large σ, is stretched vertically towards the well-defined vertical beam of non-zero velocity which we expect above the source when $\sigma = N$. A similar remark may be made concerning the horizontal velocity in the equatorial plane of a distant source in a rotating but non-stratified fluid.

We summarize the properties of the flow far from the source in an unbounded fluid as they vary with source frequency. If the fluid is stratified but not rotating ($f^2 = 0$, $N^2 > 0$), very slow source frequencies result in motions which are confined between the two horizontal planes $z = \pm a$ tangent to the source at its poles. In the present cylindrically symmetric case, the persistent motion is entirely in the radial direction, but of course a non-zero component of azimuthal velocity would be possible in the non-symmetric case. The source must continue to do work on this layer of fluid, for the motion is composed of waves of non-zero group velocity and continuously carries energy away from the source. Motions for which $0 < \sigma < N$ are confined almost entirely to the region between the two cones parallel to $\theta = \arccos \sigma/N$ but tangent to the source above and below. Again, the source must continuously do work to maintain the motion. When $\sigma = N$, the motion is confined to the interior of the vertical cylinder $r < a$ and has the structure in space of the Taylor column which the same source, acting at zero frequency in a rotating non-stratified fluid, would produce. But, in contrast to the Taylor column case, the motion at $\sigma = N$ is evanescent and requires no work by the source for its maintainence. When σ is slightly greater than N, the periodic motion is no longer confined within this sharply defined region. It is largest directly above the source but now varies smoothly with colatitude.

Finally, as σ becomes much greater than N, the stratification has an ever smaller effect on the motion and, in the limit $\sigma/N \to \infty$, the motion is simply that due to a spherical source in an unstratified perfect fluid. No work by the source is required to maintain the motion when $\sigma > N$.

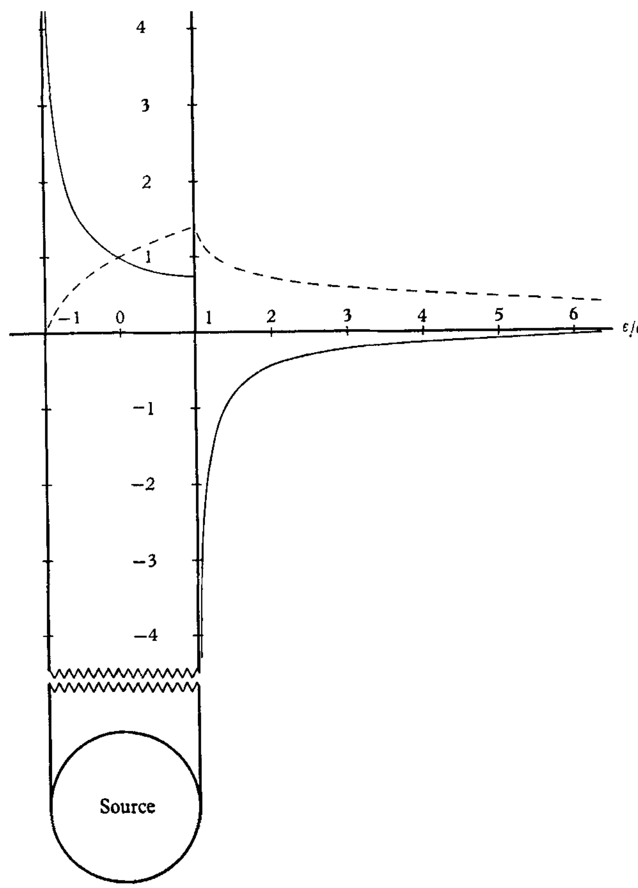

FIGURE 3. Variation in direction perpendicular to cone $\theta = \theta_\sigma$ of sin (σt) component of radial velocity (solid line) and normal velocity (dashed line) according to (42). Radial velocity is in units of

$$\left(\frac{a}{R}\right)^{\frac{1}{2}} \frac{(f^2-\sigma^2)(\sigma^2-N^2)^{\frac{1}{2}}}{(f^2-N^2)^{\frac{3}{4}}(\sin 2\theta_\sigma)^{\frac{3}{2}}}$$

with normal velocity in units of (a/R) times this quantity. The variation of the cos (σt) component is obtained by reflecting this plot through $\epsilon/a = 0$.

The motions produced by a source in a rotating but non-stratified fluid ($N^2 = 0, f^2 > 0$) are very similar. When the source frequency is very slow, the motion is in a Taylor column confined to the vertical cylinder $r < a$. The source continues to do work to maintain this motion in an unbounded fluid. If $0 < \sigma < f$, motion is confined to the vicinity of the two cones parallel to $\theta = \arcsin \sigma/f$ but tangent to the source above and below. When $\sigma = f$, the motion is entirely horizontal, between the planes tangent to the source at its poles and carries no energy away from the source. For σ slightly greater than f, the motion remains primarily horizontal but is no longer confined within a well-defined region. Finally, as $\sigma/f \to \infty$, the motion approaches that to be expected in the non-rotating perfect fluid case.

The motion when both rotation and stratification are present resembles either the first case or the second, depending upon whether $N > f$ or $f > N$, for frequencies greater than the smaller of N or f. For lower frequencies, including zero, the flow is again an appropriately stretched potential flow carrying no energy away from the source.

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